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An integral-transform method for shock–shock interaction studies

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An analytic solution is obtained for the head-on collision of strong shock waves with supersonically moving, axisymmetric, slender bodies. A new method of the solution is developed using integral transforms. Pressure distributions on the surface of a cone illustrate the effects of shock strength and body speed. The results are compared with those of Blankenship (1965), which were obtained by numerical methods.

It is also shown that the same analysis can, with certain modifications, be adapted for treating the shock-shock interaction on thin two-dimensional aerofoils of arbitrary shape at small incidence.

1. Introduction

Smyrl (1963) studied the impact of a plane shock wave of arbitrary strength on a thin two-dimensional wedge, moving at supersonic speed, by the conical-flow technique of Lighthill (1949). The aerofoil has a weak attached shock and a collision between two shocks is involved. This has been termed a shock–shock interaction. Blankenship (1965) uses an approach similar to Smyrl's in order to treat the shock–shock interaction of strong blasts on a slender supersonic cone, but he solved the boundary-value problem by numerical methods.

The purpose of this paper is to obtain an analytic solution in closed form for the head-on collision of a plane shock of arbitrary strength on supersonically moving, axisymmetric, slender bodies. The formulation of the problem is essentially the same as that of Ting & Ludloff (1952) and Ludloff & Friedman (1952) for the diffraction of blasts by stationary bodies. A new procedure is developed for the solution using integral transforms, which is simple as well as straightforward. The same method, with certain modifications, is applied to the problem of shock–shock interaction on two-dimensional, supersonic, thin aerofoils of arbitrary shape. More general results are obtained than those of Smyrl (1963), in the sense that no need exists to represent the aerofoil as a superposition of wedges, or cone fields.

The physical picture of the flow field which results after the body penetrates the shock has been discussed by Smyrl (1963), Blankenship (1965) and Blankenship & Merritt (1966). We shall summarize the essential features. We consider a plane shock of arbitrary strength (or a blast) moving freely at supersonic speed V into a gas at rest imparting a uniform velocity U to the fluid behind it and striking an axisymmetric slender body of infinite length moving in the opposite direction with supersonic speed W. If the origin for time t is so chosen that at $t = 0$ the shock front coincides with the nose of the body, there are three flow regions (0), (1) and (2) for $t \leqslant 0$ (see figure 1). In region (0) the gas is at rest, region (1) is that of uniform flow behind the plane shock and region (2) is a spatially non-uniform region. The regions (0) and (2) are separated by a weak shock or Mach-wave emanating from the apex of the body.

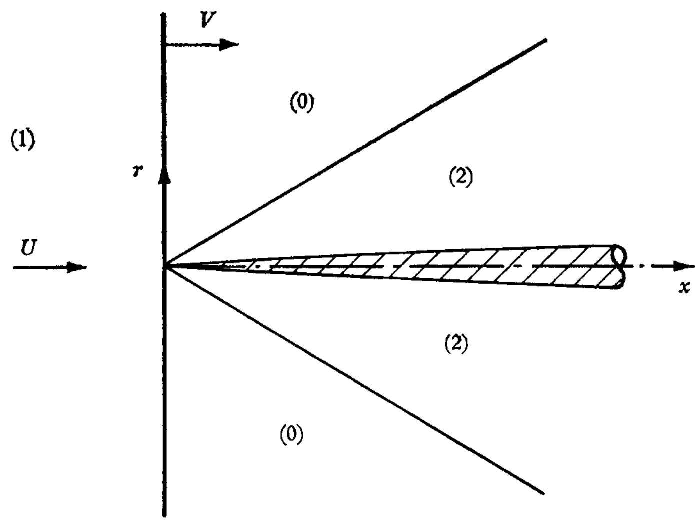

FIGURE 1. Flow pattern for $t = 0$.

For $t > 0$ the body penetrates the blast; the axisymmetric flow pattern developed is illustrated in figure 2. We choose a co-ordinate system (x, r) with origin 0 which is fixed relative to the undisturbed flow behind the shock. The presence of the body in region (1) causes a small disturbance; a spherical wavelet $BCDE$ is spread with the speed of sound a_1 with centre at 0, radius $a_1 t$, together with an attached Mach-wave AC (due to the supersonic flow $W + U > a_1$ over the body) tangent to it and the shock front. Due to interaction with the Mach-wave, the plane blast is deflected at point I along IB. The pressure in the vicinity of the body surface behind the shock influences the speed and the curvature of the shock front which in turn changes the pressure on the body. The diffracted shock is considered to be slightly deflected from the undiffracted position and meets the body surface normally to ensure that flow across the shock will remain parallel to the surface. The Mach-wave at I is also deflected and takes the position ID tangent to the reflected wave $BCDE$.

It may be pointed out here that the procedure used in this paper does not depend much on this picture, since the problem is posed generally in terms of initial and boundary values, and the solution is obtained for the entire field behind the shock. The foregoing flow pattern, however, can be seen to emerge from the solution.

The flow is considered to be inviscid and adiabatic. With regard to the flow pattern (figure 2), the region (2) is truncated but otherwise not affected by the traversing shock. The disturbance field (assumed weak) in this region is deduced on the assumption that the changes in the state of the gas are not only adiabatic but isentropic too. In the disturbed region behind the shock this supposition is untenable, since the air enters this region across a curved shock and we expect rotational flow. The downstream perturbations will be weak compared with the undisturbed flow of region (1). Thus a linear treatment of the flow field behind the shock based on region (1) is permissible.

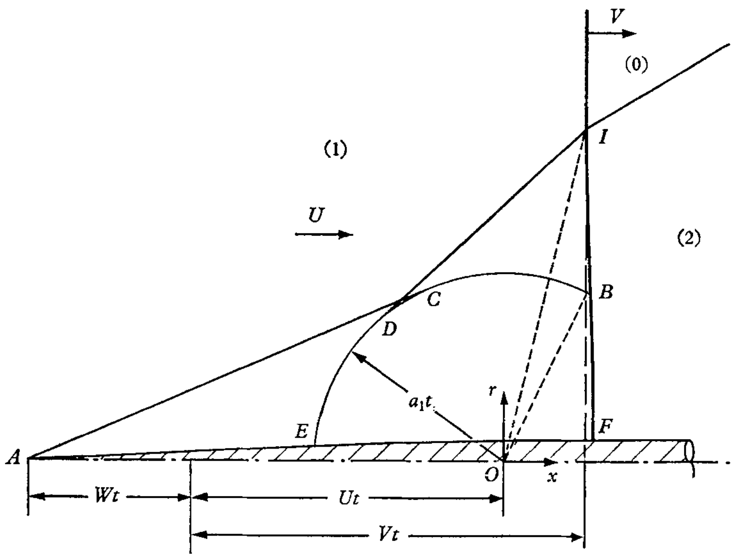

FIGURE 2. Flow pattern for $t > 0$.

The physical quantities defining the problem are $M = V/a_0$ the Mach number of the shock, $M' = W/a_0$ the Mach number of the body and $r = f(\chi)$ the function defining the surface of the body, where χ denotes the axial co-ordinate with origin at the nose of the body.

Undisturbed shock propagation

If R_0, P_0, a_0 and R_1, P_1, a_1 be the density, pressure and speed of sound of the undisturbed flow ahead of and behind the shock, the conservation of mass, momentum and energy together with the equation of state applied across the shock give

$$\left.\begin{aligned} R_1/R_0 &= (\gamma+1)M^2/\{(\gamma-1)M^2+2\},\\ P_1/P_0 &= \{2\gamma M^2-(\gamma-1)\}/(\gamma+1),\\ a_1/a_0 &= [\{2\gamma M^2-(\gamma-1)\}\{(\gamma-1)M^2+2\}]^{\frac{1}{2}}/(\gamma+1)M,\\ M_1 = U/a_1 &= 2(M^2-1)/[\{2\gamma M^2-(\gamma-1)\}\{(\gamma-1)M^2+2\}]^{\frac{1}{2}}. \end{aligned}\right\} \quad (1.1)$$

2. Disturbance field behind the shock

The equations governing the unsteady rotational flow behind the shock are

$$\text{continuity} \quad \partial R/\partial t + \nabla . (R\mathbf{V}) = 0, \tag{2.1}$$

$$\text{momentum} \quad \partial \mathbf{V}/\partial t + (\mathbf{V}.\nabla)\mathbf{V} = -(1/R)\nabla P, \tag{2.2}$$

$$\text{energy} \quad c_v R\{\partial\theta/\partial t + (\mathbf{V}.\nabla)\theta\} = -P(\nabla.\mathbf{V}), \tag{2.3}$$

$$\text{state} \quad P = JR\theta. \tag{2.4}$$

Equations (2.1), (2.3) and (2.4) can be combined to yield the adiabatic relation

$$(\partial/\partial t + \mathbf{V}.\nabla)(PR^{-\gamma}) = 0, \tag{2.5}$$

where R, P, θ, $\mathbf{V}$ denote the total values for density, pressure, temperature and velocity vector of the disturbed flow behind the shock, J is the gas constant and $\gamma = c_p/c_v$ is the ratio of the specific heats.

Noting that the velocity of the undisturbed flow in region (1) is zero for the chosen co-ordinate system and assuming that behind the shock R, P differ only by small quantities from the undisturbed flow values R_1, P_1, the equations for the disturbance field behind the shock for axisymmetric flow (linearizing (2.1), (2.2) and (2.5)) will be given by

$$\left.\begin{aligned} &\partial\rho_1/\partial t + R_1\,\partial u_1/\partial x + R_1(\partial q_1/\partial r + q_1/r) = 0,\\ &\partial u_1/\partial t = -(1/R_1)\,\partial p_1/\partial x, \quad \partial q_1/\partial t = -(1/R_1)\,\partial p_1/\partial r,\\ &\partial p_1/\partial t = (\gamma P_1/R_1)\,\partial\rho_1/\partial t, \end{aligned}\right\} \tag{2.6}$$

where ρ_1, p_1 and (u_1, q_1) denote the perturbation density, pressure and the axial and radial disturbance velocities behind the shock. These parameters can be expressed in non-dimensional form as $\rho_1/R_1 = \rho$, $p_1/\gamma P_1 = p$, $u_1/a_1 = u$, $q_1/a_1 = q$ and replacing the time variable t by a reduced space variable $\tau = a_1 t$, the set of equations (2.6) becomes

$$\partial\rho/\partial\tau + \partial u/\partial x + (\partial q/\partial r + q/r) = 0, \tag{2.7}$$

$$\partial u/\partial\tau = -\partial p/\partial x, \quad \partial q/\partial\tau = -\partial p/\partial r, \tag{2.8}$$

$$\partial p/\partial\tau = \partial\rho/\partial\tau. \tag{2.9}$$

These can be combined to yield

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} - \frac{\partial^2 p}{\partial\tau^2} = 0, \tag{2.10}$$

which is a wave equation for p, the non-dimensional perturbation pressure behind the shock.

3. The initial and boundary conditions

The initial conditions are

$$p(x,r,\tau) = \partial p(x,r,\tau)/\partial \tau = 0, \quad \text{for} \quad \tau \leqslant 0. \tag{3.1}$$

On the disturbed shock front

The disturbed shock may be regarded as behaving in a quasi-steady manner, i.e. in a co-ordinate system fixed in the undisturbed shock itself, the downstream perturbation parameters ρ_1, p_1, u_1, q_1 are related to the upstream perturbation parameters ρ_2, p_2, u_2, q_2 at the shock through the usual shock relations. To arrive at the specific results, a local element of the disturbed shock front is isolated as

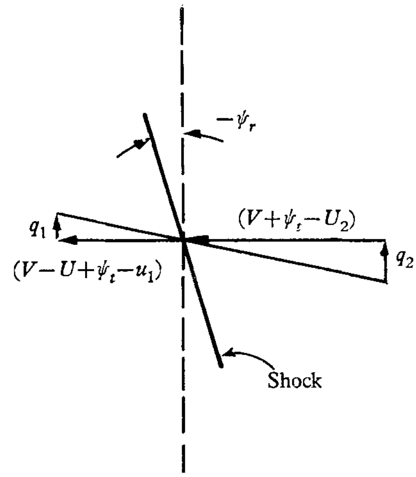

FIGURE 3. Local conditions at the perturbed shock.

shown in figure 3. The unsteady displacement of the shock from its undisturbed position is denoted by $\psi(r,t)$ and is assumed to be small. The shock velocity is directed normal to the shock front and has components in the x and r directions which are $V+\partial\psi/\partial t$ and $-V\partial\psi/\partial r$ with respect to the upstream flow. These quantities are substituted into the Rankine–Hugoniot relations which are then linearized. The result is a set of simultaneous equations for ρ_1, p_1, u_1, q_1 in terms of ρ_2, p_2, u_2, q_2 and derivatives of ψ. The procedure has been carried out by Moore (1953) and the following relations can thus be found

$$\left.\begin{aligned} \frac{\psi_t-u_1}{V} &= B_1\frac{\psi_t-u_2}{V}+B_2\frac{p_2}{P_0}, & \frac{\rho_1}{R_1} &= C_1\frac{\psi_t-u_2}{V}+C_2\frac{p_2}{P_0}, \\ \frac{q_1}{V} &= \frac{q_2}{V}-\frac{U}{V}\psi_r, & \frac{p_1}{P_1} &= D_1\frac{\psi_t-u_2}{V}+D_2\frac{p_2}{P_0}, \end{aligned}\right\} \tag{3.2}$$

where

$$\begin{aligned} B_1 &= \{(\gamma-1)-2/M^2\}/(\gamma+1), & B_2 &= 2(\gamma-1)/\gamma(\gamma+1)M^2, \\ C_1 &= 4/\{(\gamma-1)M^2+2\}, & C_2 &= (1-[2(\gamma-1)/\{(\gamma-1)M^2+2\}])/\gamma, \\ D_1 &= 4\gamma M^2/\{2\gamma M^2-(\gamma-1)\}, & D_2 &= \{2M^2-(\gamma-1)\}/\{2\gamma M^2-(\gamma-1)\}. \end{aligned}$$

The perturbation parameters behind the shock may be expressed in non-dimensional form as before. Ahead of the shock we write $p_2/\gamma P_0 = \bar{p}$, $u_2/V = \bar{u}$, $q_2/V = \bar{q}$. The relations (3.2) may then be put in the form

$$\rho = \Lambda_{11}\bar{u}+\Lambda_{12}\bar{p}+\pi_{11}\psi_\tau, \tag{3.3a}$$

$$p = \Lambda_{21}\bar{u}+\Lambda_{22}\bar{p}+\pi_{21}\psi_\tau, \tag{3.3b}$$

$$u = \Lambda_{31}\bar{u}+\Lambda_{32}\bar{p}+\pi_{31}\psi_\tau, \tag{3.3c}$$

$$q = \Lambda_{41}\bar{q}+\pi_{41}\psi_r, \tag{3.3d}$$

where

$$\Lambda_{11} = -C_1, \quad \Lambda_{12} = \gamma C_2, \quad \pi_{11} = C_1(a_1/a_0)/M,$$

$$\Lambda_{21} = -D_1/\gamma, \quad \Lambda_{22} = D_2, \quad \pi_{21} = D_1(a_1/a_0)/\gamma M,$$

$$\Lambda_{31} = B_1(a_0/a_1)M, \quad \Lambda_{32} = -\gamma B_2(a_0/a_1)M, \quad \pi_{31} = (1-B_1),$$

$$\Lambda_{41} = (a_0/a_1)M, \quad \pi_{41} = -M_1.$$

The shock relations (3.3) will be applied at $x = m\tau$, where

$$m = (V-U)/a_1 = M(a_0/a_1)-M_1,$$

the undeflected plane of the shock in accordance with the linearization.

From shock relations (3.3*b*) and (3.3*c*) we eliminate ψ_τ and from (3.3*b*) and (3.3*d*) we eliminate ψ by cross-differentiation and obtain at $x = m\tau$

$$u = (1/A)(p-B\bar{u}-C\bar{p}), \tag{3.4a}$$

and

$$\frac{\partial q}{\partial \tau} = \frac{1}{D}\left(\frac{\partial p}{\partial r}-\Lambda_{21}\frac{\partial \bar{u}}{\partial r}-\Lambda_{22}\frac{\partial \bar{p}}{\partial r}\right)+\Lambda_{41}\frac{\partial \bar{q}}{\partial \tau}, \tag{3.4b}$$

where

$$A = \pi_{21}/\pi_{31}, \quad B = \Lambda_{21}-A\Lambda_{31}, \quad C = \Lambda_{22}-A\Lambda_{32} \quad \text{and} \quad D = \pi_{21}/\pi_{41}.$$

On the body surface

Owing to singularities at $r = 0$, it is necessary to formulate the linearized boundary condition at the body surface carefully. With respect to the co-ordinates attached to the body, if $r = f(\chi)$ represents the cross-sectional radius of a body moving with speed W, the tangency condition that the normal component of the velocity vanishes along the surface may be expressed as

$$q_2/(W+u_2) = f'(\chi) \approx q_2/W,$$

where the prime represents the differentiation with respect to the argument. Assuming q_2 inversely proportional to r in the vicinity of the axis we obtain there $rq_2 = Wf(\chi)f'(\chi)$. Hence in the assumed co-ordinate system (x, r, t) behind the shock we can write, near the body surface,

$$rq_1 = (W+U)f\{x+(W+U)t\}f'\{x+(W+U)t\},$$

or in non-dimensional form

$$rq = m_1 f(x+m_1\tau)f'(x+m_1\tau), \tag{3.5}$$

where $m_1 = (W+U)/a_1 = M'(a_0/a_1)+M_1$. Using the second momentum equation (2.8) it follows that as $r \to 0$,

$$r(\partial p/\partial r) = -\partial(rq)/\partial\tau \to -m_1^2 F(x+m_1\tau), \tag{3.6}$$

where $F(\xi) = f'^2(\xi)+f(\xi)f''(\xi)$.

Again, on the body surface along the shock $x = m\tau$

$$rq = m_1 f\{(m+m_1)\tau\}f'\{(m+m_1)\tau\}.$$

Using the shock relation (3.4*b*), it follows that along the shock as $r \to 0$,

$$r(\partial p/\partial r) \to Dm_1(m+m_1)F\{(m+m_1)\tau\}+\lim_{r\to 0}\{\Lambda_{21}r(\partial\bar{u}/\partial r)+\Lambda_{22}r(\partial\bar{p}/\partial r) - D\Lambda_{41}r(\partial\bar{q}/\partial\tau)\}. \tag{3.7}$$

Hence the conditions at the shock (3.4) should be supplemented by condition (3.7) to ensure that the flow remains tangential to the body surface at the foot of the shock.

At infinity

We prescribe that all perturbations vanish at infinity, i.e.

$$\text{as} \quad x \to -\infty, \quad r \to \infty, \quad p \text{ and its derivatives} \to 0. \tag{3.8}$$

4. Disturbance field ahead of shock

We assume that the traversing shock does not affect the flow field ahead of it. Hence time-independent solutions can be found for the region (2). For flow over an axisymmetric body with a supersonic free-stream speed $W > a_0$, the perturbation velocity potential $\phi(\chi, r)$ is given by

$$\phi(\chi,r) = -W\int_0^{(\chi-\beta r)} \frac{f(\xi)f'(\xi)}{\{(\chi-\xi)^2-\beta^2r^2\}^{\frac{1}{2}}}\,d\xi, \tag{4.1}$$

where $\beta^2 = (W/a_0)^2-1$, with respect to the axes attached to the nose of the body. The perturbation velocities are

$$\left.\begin{aligned} u_2 &= \partial\phi/\partial\chi = -W\int_0^{(\chi-\beta r)} \frac{F(\xi)}{\{(x-\xi)^2-\beta^2r^2\}^{\frac{1}{2}}}\,d\xi, \\ q_2 &= \partial\phi/\partial r = \frac{W}{r}\int_0^{(\chi-\beta r)} \frac{(\chi-\xi)F(\xi)}{\{(\chi-\xi)^2-\beta^2r^2\}^{\frac{1}{2}}}\,d\xi, \end{aligned}\right\} \tag{4.2}$$

where $F(\xi)$ is as defined in (3.6).

For the pressure we use the quadratic approximation to the Bernoulli's equation, i.e. $p_2/R_0 = -W(\partial\phi/\partial\chi)-\frac{1}{2}(\partial\phi/\partial r)^2$. The second term is negligible except in the vicinity of the surface of the body where the order of $\partial\phi/\partial r$ is different from that of $\partial\phi/\partial\chi$. We shall consistently neglect the contribution of $(\partial\phi/\partial r)^2$ to p_2 and consider the approximation $p_2 = -R_0W(\partial\phi/\partial\chi)$.

With respect to the co-ordinate system (x, r, t), χ is replaced by $(x+m_1\tau)$. Hence at the shock, where $x = m\tau$, the perturbation quantities in the non-dimensional form may be expressed as

$$\left.\begin{aligned} \bar{u} &= -k_1 \int_0^{\{(m+m_1)\tau-\beta r\}} \frac{F(\xi)}{[\{(m+m_1)\tau-\xi\}^2-\beta^2 r^2]^{\frac{1}{2}}}\,d\xi, \\ \bar{q} &= \frac{k_1}{r}\int_0^{\{(m+m_1)\tau-\beta r\}} \frac{\{(m+m_1)\tau-\xi\}F(\xi)}{[\{(m+m_1)\tau-\xi\}^2-\beta^2 r^2]^{\frac{1}{2}}}\,d\xi, \\ \bar{p} &= k_2(-\bar{u}/k_1), \end{aligned}\right\} \tag{4.3}$$

where $k_1 = W/V = M'/M$ and $k_2 = (R_0/\gamma P_0)\,W^2 = M'^2$.

5. The Lorentz transformation

We introduce new independent variables $(\bar{x}, r, \bar{\tau})$ related to the variables (x, r, τ) by the Lorentz transformation

$$\bar{x} = (x-m\tau)/(1-m^2)^{\frac{1}{2}}, \quad \bar{\tau} = (\tau-mx)/(1-m^2)^{\frac{1}{2}}, \tag{5.1}$$

such that the plane $\bar{x} = 0$ corresponds to the shock plane $x = m\tau$, and the wave equation (2.10) remains invariant, viz.

$$\frac{\partial^2 p}{\partial \bar{x}^2}+\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}-\frac{\partial^2 p}{\partial \bar{\tau}^2} = 0. \tag{5.2}$$

The initial conditions (3.1) and boundary conditions (3.8) and (3.6) become for

$$\bar{\tau} \leqslant 0, \quad p(\bar{x}, r, \bar{\tau}) = \partial p(\bar{x}, r, \bar{\tau})/\partial \bar{\tau} = 0, \tag{5.3}$$

$$\bar{\tau} > 0, \quad \bar{x} < 0, \quad \text{as} \quad \bar{x} \to -\infty, \quad r \to \infty, \quad p \text{ and its derivatives} \to 0, \tag{5.4}$$

$$\bar{\tau} > 0, \quad \bar{x} < 0, \quad \text{as} \quad r \to 0, \quad r(\partial p/\partial r) \to A_0 F\{\bar{a}(\bar{\tau}+\lambda_0 \bar{x})\}, \tag{5.5}$$

where $A_0 = -m_1^2$, $\bar{a} = (m+m_1)/(1-m^2)^{\frac{1}{2}}$ and $\lambda_0 = (1+mm_1)/(m+m_1)$.

With the Lorentz transformation (2.7), combined with (2.9) and the first of equations (2.8), give

$$(\partial/\partial\bar{\tau} - m\,\partial/\partial\bar{x})p + (\partial/\partial\bar{x} - m\,\partial/\partial\bar{\tau})u + (1-m^2)^{\frac{1}{2}}(\partial/\partial r + 1/r)q = 0,$$

and

$$m\,\partial u/\partial\bar{x} = \partial u/\partial\bar{\tau} + (\partial/\partial\bar{x} - m\,\partial/\partial\bar{\tau})p.$$

Eliminating $\partial u/\partial\bar{x}$ and differentiating with respect to $\bar{\tau}$ it follows that

$$\frac{\partial^2 p}{\partial\bar{x}\,\partial\bar{\tau}}+\frac{\partial^2 u}{\partial\bar{\tau}^2}+\frac{m}{(1-m^2)^{\frac{1}{2}}}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial q}{\partial\bar{\tau}} = 0. \tag{5.6}$$

From the shock relations (3.4), substituting for u and $\partial q/\partial\bar{\tau} = (\partial q/\partial\tau)/(1-m^2)^{\frac{1}{2}}$ in (5.6) we obtain at the shock $\bar{x} = 0$

$$\begin{aligned} &\frac{\partial^2 p}{\partial\bar{x}\,\partial\bar{\tau}}+\frac{1}{A}\frac{\partial^2 p}{\partial\bar{\tau}^2}+\frac{m}{1-m^2}\frac{1}{D}\left(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}\right) \\ &\qquad = \frac{1}{A}\frac{\partial^2}{\partial\bar{\tau}^2}(B\bar{u}+C\bar{p})+\frac{m}{1-m^2}\frac{1}{D}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)(\Lambda_{21}\bar{u}+\Lambda_{22}\bar{p}) \\ &\qquad\qquad -\frac{m}{(1-m^2)^{\frac{1}{2}}}\Lambda_{41}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial\bar{q}}{\partial\bar{\tau}}. \end{aligned} \tag{5.7}$$

From (4.3) the perturbation quantities $\bar{u}$, $\bar{q}$, $\bar{p}$ at the shock in terms of new variables are

$$\left.\begin{aligned} \bar{u} &= -k_1 \int_0^{\bar{\tau}-\bar{b}r} \frac{F(\bar{a}\mu)}{\{(\bar{\tau}-\mu)^2-\bar{b}^2r^2\}^{\frac{1}{2}}}\, d\mu, \\ \bar{q} &= k_1 \frac{\bar{a}}{r} \int_0^{\bar{\tau}-\bar{b}r} \frac{(\bar{\tau}-\mu)F(\bar{a}\mu)}{\{(\bar{\tau}-\mu)^2-\bar{b}^2r^2\}^{\frac{1}{2}}}\, d\mu, \\ \bar{p} &= k_2(-\bar{u}/k_1), \end{aligned}\right\} \tag{5.8}$$

where $\bar{b} = \beta/\bar{a}$. Substituting for $\bar{u}$, $\bar{q}$ and $\bar{p}$ from (5.8) in (5.7) and simplifying, we obtain at $\bar{x} = 0$

$$-\left(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}\right)+2m\frac{\partial^2 p}{\partial \bar{x}\,\partial \bar{\tau}}+\left(1+\frac{1}{M^2}\right)\frac{\partial^2 p}{\partial \bar{\tau}^2} = K\frac{\partial^2}{\partial \bar{\tau}^2}\int_0^{\bar{\tau}-\bar{b}r}\frac{F(\bar{a}\mu)}{\{(\bar{\tau}-\mu)^2-\bar{b}^2r^2\}^{\frac{1}{2}}}\,d\mu, \tag{5.9}$$

where K is a constant given by

$$K = -(1+1/M^2)(Bk_1-Ck_2)+(\Lambda_{21}k_1-\Lambda_{22}k_2)\bar{b}^2-(1-m^2)^{\frac{1}{2}}D\Lambda_{41}k_1\bar{a}\bar{b}^2.$$

Equation (5.9) is thus a second-order differential condition at the shock $\bar{x} = 0$ in terms of p.

Also making use of (5.8) in (3.7) we deduce that

$$\text{at}\quad \bar{x} = 0,\quad \text{as}\quad r \to 0,\quad r(\partial p/\partial r) \to B_0 F(\bar{a}\,\bar{\tau}), \tag{5.10}$$

where B_0 is another constant given by

$$B_0 = (1-m^2)^{\frac{1}{2}}D(m_1-\Lambda_{41}k_1)\bar{a}+(\Lambda_{21}k_1-\Lambda_{22}k_2).$$

Hence the wave equation (5.2) is to be solved subject to the initial conditions (5.3) and boundary conditions (5.4), (5.5), (5.9) and (5.10).

6. Analytic solution

The solution to the above formulation is sought by means of integral transforms. First we apply the Laplace transform with respect to $\bar{\tau}$ and then the Hankel transform with respect to r (see Sneddon 1951)

$$v(\bar{x}, r, s) = L\{p(\bar{x}, r, \bar{\tau})\} = \int_0^\infty p(\bar{x}, r, \bar{\tau}) \exp\{-s\bar{\tau}\}\, d\bar{\tau}, \tag{6.1}$$

and

$$w(\bar{x}, \alpha, s) = H\{v(\bar{x}, r, s)\} = \int_0^\infty r v(\bar{x}, r, s)\, J_0(\alpha r)\, dr. \tag{6.2}$$

Application of Laplace transform to the wave equation (5.2) together with the initial conditions (5.3) gives

$$\frac{\partial^2 v}{\partial \bar{x}^2}+\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-s^2 v = 0. \tag{6.3}$$

The boundary conditions (5.4), (5.5), (5.9) and (5.10) reduce to

$$\text{for} \quad \bar{x} < 0, \quad \text{as} \quad \bar{x} \to -\infty, \quad r \to \infty, \quad v \text{ and its derivatives} \to 0, \tag{6.4}$$

$$\bar{x} < 0, \quad \text{as} \quad r \to 0, \quad r(\partial v/\partial r) = A_0 \exp\{s\lambda_0 \bar{x}\} G(s), \tag{6.5}$$

$$\text{at} \quad \bar{x} = 0, \quad r > 0, \quad -\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) + 2ms\frac{\partial v}{\partial x} + (1 + 1/M^2)s^2 v = Ks^2 G(s) K_0(\bar{b}rs), \tag{6.6}$$

$$\bar{x} = 0, \quad \text{as} \quad r \to 0, \quad r(\partial v/\partial r) = B_0 G(s), \tag{6.7}$$

where $G(s) = L\{F(\bar{a}\bar{r})\}$ and $K_0(\bar{b}rs)$ is the modified Bessel function of the second kind.

Next, applying the Hankel transform to equation (6.3), together with the conditions (6.4) and (6.5), yields

$$\frac{\partial^2 w}{\partial \bar{x}^2} - \lambda^2 w = A_0 \exp\{s\lambda_0 \bar{x}\} G(s), \tag{6.8}$$

and

$$\text{as} \quad \bar{x} \to -\infty, \quad w \to 0. \tag{6.9}$$

The shock condition (6.6), together with (6.7), gives at $\bar{x} = 0$

$$\left(\lambda^2 + \frac{s^2}{M^2}\right) w + 2ms\frac{\partial w}{\partial \bar{x}} = -B_0 G(s) + K\frac{s^2}{\alpha^2 + \bar{b}^2 s^2} G(s), \tag{6.10}$$

where $\lambda^2 = \alpha^2 + s^2$.

Hence the problem is reduced to the solution of (6.8) subject to the boundary conditions (6.9) and (6.10). The complete solution of (6.8) may be written as

$$w = E_1 \exp\{\lambda\bar{x}\} + E_2 \exp\{-\lambda\bar{x}\} - \frac{A_0}{\lambda^2 - \lambda_0^2 s^2} \exp\{s\lambda_0 \bar{x}\} G(s). \tag{6.11}$$

In view of the condition (6.9), the coefficient of $\exp\{-\lambda\bar{x}\}$ must vanish and E_1 is determined by using condition (6.10). Hence we obtain

$$w = \left(A_0 \frac{\lambda}{\lambda^2 - \lambda_0^2 s^2} - 2mA_0 \frac{\lambda}{H(\lambda)} \frac{s}{\lambda + \lambda_0 s} - B_0 \frac{\lambda}{H(\lambda)} + K\frac{\lambda}{H(\lambda)} \frac{s^2}{\alpha^2 + \bar{b}^2 s^2}\right) \times \frac{\exp\{\lambda\bar{x}\}}{\lambda} G(s) - \frac{A_0}{\lambda^2 - \lambda_0^2 s^2} \exp\{s\lambda_0 \bar{x}\} G(s), \tag{6.12}$$

where $H(\lambda) = \lambda^2 + 2ms\lambda + (s^2/M^2) = (\lambda + \lambda_2 s)(\lambda + \lambda_3 s)$ with λ_2, λ_3 the roots (real, distinct and positive) of the quadratic equation $\lambda_i^2 - 2m\lambda_i + (1/M^2) = 0$. Then (6.12) may be written in the form

$$w = \tfrac{1}{2}A_0 \left(\frac{1}{\lambda - \lambda_0 s} \frac{\exp\{\lambda\bar{x}\}}{\lambda} - \frac{2}{\lambda^2 - \lambda_0^2 s^2} \exp\{s\lambda_0 \bar{x}\}\right) G(s) + \frac{1}{2}\sum_{i=1}^{5} A_i \frac{1}{\lambda + \lambda_i s} \frac{\exp\{\lambda\bar{x}\}}{\lambda} G(s), \tag{6.13}$$

where

$$A_1 = A_0 H(\lambda_0)/H(-\lambda_0), \quad H(\lambda_0) = \lambda_0^2 + 2m\lambda_0 + (1/M^2),$$

$$A_2 = -\frac{2\lambda_2}{\lambda_2 - \lambda_3}\left(B_0 + \frac{2mA_0}{\lambda_0 - \lambda_2} - \frac{K}{\lambda_2^2 - \lambda_4^2}\right),$$

$$A_3 = \frac{2\lambda_3}{\lambda_2 - \lambda_3}\left(B_0 + \frac{2mA_0}{\lambda_0 - \lambda_3} - \frac{K}{\lambda_3^2 - \lambda_4^2}\right),$$

$$A_4 = K/(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4), \quad A_5 = K/(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4),$$

$$\lambda_1 = \lambda_0, \quad \lambda_5 = -\lambda_4 \quad \text{and} \quad \lambda_4^2 = (1 - \bar{b}^2) > 0.$$

The expression $\lambda_4^2 > 0$ implies that the intersection point I of the Mach-wave attached to the body with the plane blast lies outside the sonic circle (figure 2). By further manipulation (6.13) may be expressed as

$$w(\bar{x}, \alpha, s) = \frac{1}{2}\left(-A_0 \int_{-\infty}^{0} \exp\{s\lambda_0 \xi\} G(s) \frac{\exp\{-\lambda(\bar{x}-\xi)\}}{\lambda} d\xi \right.$$
$$\left. + \sum_{i=1}^{5} A_i \int_{0}^{\infty} \exp\{-s\lambda_i \xi\} G(s) \frac{\exp\{-\lambda(\xi-\bar{x})\}}{\lambda} d\xi\right). \quad (6.14)$$

Finally, we seek inversion of $w(\bar{x}, \alpha, s)$ to obtain $p(\bar{x}, r, \bar{\tau})$, i.e.

$$p(\bar{x}, r, \bar{\tau}) = L^{-1}[H^{-1}\{w(\bar{x}, \alpha, s)\}].$$

The inversion is achieved in a straightforward manner with the aid of standard tables of integral transforms. Thus

$$L^{-1}\left\{H^{-1}\left(\int_{-\infty}^{0} \exp\{s\lambda_0\xi\} G(s) \frac{\exp\{-\lambda(\bar{x}-\xi)\}}{\lambda} d\xi\right)\right\}$$
$$= L^{-1}\left(\int_{-\infty}^{0} \exp\{s\lambda_0\xi\} G(s) \frac{\exp[-s\{(\bar{x}-\xi)^2+r^2\}^{\frac{1}{2}}]}{\{(\bar{x}-\xi)^2+r^2\}^{\frac{1}{2}}} d\xi\right)$$
$$= \int_{-\infty}^{0} \frac{F\{\bar{a}(\bar{\tau}-R+\lambda_0\xi)\}}{R} d\xi,$$

where $R = \{(\bar{x}-\xi)^2+r^2\}^{\frac{1}{2}}$. Similarly, the inversion for the other expression in (6.14) can be achieved and we obtain

$$p(\bar{x}, r, \bar{\tau}) = \frac{1}{2}\left(-A_0 \int_{-\infty}^{0} \frac{F\{\bar{a}(\bar{\tau}-R+\lambda_0\xi)\}}{R} d\xi \right.$$
$$\left. + \sum_{i=1}^{5} A_i \int_{0}^{\infty} \frac{F\{\bar{a}(\bar{\tau}-R-\lambda_i\xi)\}}{R} d\xi\right). \quad (6.15)$$

Since the initial conditions (5.3) imply that $F(\bar{x}, r, \bar{\tau}) = 0$ for $\bar{\tau} \leqslant 0$, the last expression can also be written as

$$p(\bar{x}, r, \bar{\tau}) = \frac{1}{2}\left(-A_0 \int_{\omega_0}^{0} \frac{F\{\bar{a}(\bar{\tau}-R+\lambda_0\xi)\}}{R} d\xi \right.$$
$$\left. + \sum_{i=1}^{5} A_i \int_{0}^{\omega_i} \frac{F\{\bar{a}(\bar{\tau}-R-\lambda_i\xi)\}}{R} d\xi\right), \quad (6.16)$$

where $\omega_0 = [\bar{x}+\lambda_0\bar{\tau}-\{(\bar{\tau}+\lambda_0\bar{x})^2-(1-\lambda_0^2)r^2\}^{\frac{1}{2}}]/(1-\lambda_0^2)$

and $\omega_i = [\bar{x}-\lambda_i\bar{\tau}+\{(\bar{\tau}-\lambda_i\bar{x})^2-(1-\lambda_i^2)r^2\}^{\frac{1}{2}}]/(1-\lambda_i^2)$.

In case $\lambda_4^2 = (1-\bar{b}^2) < 0$, the shock intersection I lies inside the sonic circle (figure 4). The region *IBDI* of figure 2 now disappears. In the above expressions λ_4 and λ_5 become imaginary. Hence for obtaining $p(\bar{x}, r, \bar{\tau})$ from (6.16) we need to consider the real parts of expressions involving λ_4 and λ_5.

The solution given by (6.16) satisfies the wave equation and all the initial and boundary conditions. By using the transformation (5.1), p can be expressed in terms of the original variables (x, r, τ).

The expression (6.14) could also be written as

$$w(\bar{x},\alpha,s) = -\frac{A_0}{\lambda^2-\lambda_0^2 s^2}\exp\{s\lambda_0\bar{x}\}G(s)+\tfrac{1}{2}A_0\int_0^{\infty}\exp\{s\lambda_0\xi\}G(s)\frac{\exp\{-\lambda(\xi-\bar{x})\}}{\lambda}d\xi$$
$$+\frac{1}{2}\sum_{i=1}^{4}A_i\int_0^{\infty}\exp\{-s\lambda_i\xi\}G(s)\frac{\exp\{-\lambda(\xi-\bar{x})\}}{\lambda}d\xi$$
$$+\frac{A_5}{\lambda^2-\lambda_4^2 s^2}\exp\{s\lambda_4\bar{x}\}G(s)-\tfrac{1}{2}A_5\int_{-\infty}^{0}\exp\{s\lambda_4\xi\}G(s)\frac{\exp\{-\lambda(\bar{x}-\xi)\}}{\lambda}d\xi. \quad (6.14)$$

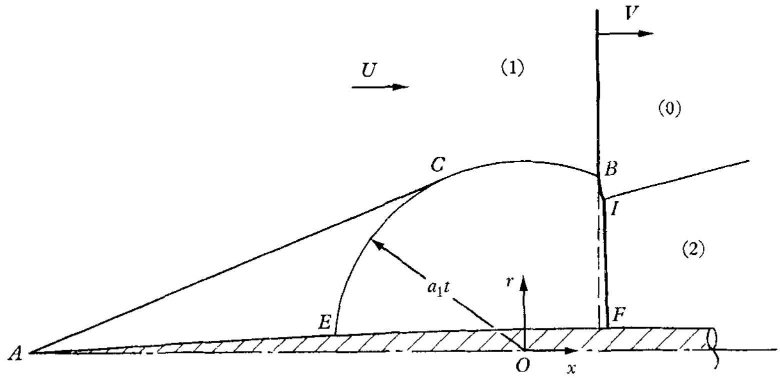

FIGURE 4. Flow pattern for $t > 0$, when I is inside the sonic circle.

Upon inversion (6.14) yields

$$p(\bar{x},r,\bar{\tau}) = -A_0\int_0^{\{\bar{\tau}+\lambda_0\bar{x}-(1-\lambda_0^2)^{\frac{1}{2}}r\}}\frac{F(\bar{a}\mu)}{\{(\bar{\tau}+\lambda_0\bar{x}-\mu)^2-(1-\lambda_0^2)r^2\}^{\frac{1}{2}}}d\mu$$
$$+\frac{1}{2}\left(A_0\int_0^{\infty}\frac{F\{\bar{a}(\bar{\tau}-R+\lambda_0\xi)\}}{R}d\xi+\sum_{i=1}^{4}A_i\int_0^{\infty}\frac{F\{\bar{a}(\bar{\tau}-R-\lambda_i\xi)\}}{R}d\xi\right.$$
$$\left.-A_5\int_{-\infty}^{0}\frac{F\{\bar{a}(\bar{\tau}-R+\lambda_4\xi)\}}{R}d\xi\right)$$
$$+A_5\int_0^{\{\bar{\tau}+\lambda_4\bar{x}-(1-\lambda_4^2)^{\frac{1}{2}}r\}}\frac{F(\bar{a}\mu)}{\{(\bar{\tau}+\lambda_4\bar{x}-\mu)^2-(1-\lambda_4^2)r^2\}^{\frac{1}{2}}}d\mu. \quad (6.15)$$

From this expression we observe that the integrals within brackets vanish for $(\bar{x}^2+r^2) \geqslant \bar{\tau}^2$, i.e. on the spherical wavelet $BCDE$ of figure 2. The first integral vanishes for $\{\bar{\tau}+\lambda_0\bar{x}-(1-\lambda_0^2)^{\frac{1}{2}}r\} \leqslant 0$, i.e. to the left of AC which is tangent to the wavelet $BCDE$ at C and passing through A, the vertex of the axisymmetric body. Finally, the last integral vanishes for $\{\bar{\tau}+\lambda_4\bar{x}-(1-\lambda_4^2)^{\frac{1}{2}}r\} \leqslant 0$, i.e. to the left of ID tangent to $BCDE$ at D and passing through I, the shock intersection. Hence the flow pattern which was discussed in § 1 can be seen to emerge from the solution.

By using the differential equations (2.8) and (2.9), the shock conditions (3.3) and the expressions (4.3), we can obtain the velocity components, the density in the disturbance field behind the shock and the form of the incident shock front. Hence the density variation may be expressed as

$$\rho(x,r,\tau) = p(x,r,\tau) + \Gamma(x,r), \tag{6.17}$$

where

$$\Gamma(x,r) = (E-1)p(x,r,\tau = x/m) - (Fk_1 - Gk_2)\int_0^{(x-\bar{c}r)} \frac{F(\beta\mu/\bar{c})}{\{(x-\mu)^2 - \bar{c}^2r^2\}^{\frac{1}{2}}}\, d\mu,$$

with

$$E = \pi_{11}/\pi_{21}, \quad F = \Lambda_{11} - E\Lambda_{21}, \quad G = \Lambda_{12} - E\Lambda_{22}, \quad \bar{c} = \beta m/(m+m_1).$$

The form of the incident shock is given by

$$x = m\tau + \psi(r,\tau), \tag{6.18}$$

where

$$\psi(r,\tau) = \frac{1}{\pi_{21}}\int_0^{\tau} p(x = m\mu, r, \mu)\, d\mu + \frac{(\Lambda_{21}k_1 - \Lambda_{22}k_2)}{\pi_{21}} \times \int_0^{\tau} d\mu \int_0^{\{(m+m_1)\mu - \beta r\}} \frac{F(\xi)}{[\{(m+m_1)\mu - \xi\}^2 - \beta^2 r^2]^{\frac{1}{2}}}\, d\xi.$$

7. Aerofoil of arbitrary shape

In the case of two-dimensional aerofoils, the flow pattern developed after the aerofoil penetrates the shock is essentially the same as discussed for axisymmetric bodies, except that now it has to be visualized in two dimensions. Since the flow at all times is supersonic with respect to the aerofoil, the flow patterns on the two sides are independent of each other. Hence it is sufficient to consider the solution for, say, $y > 0$. If (x, y, t) be the co-ordinate system fixed in the undisturbed flow behind the plane shock, the equations of motion for small perturbations in the non-dimensional form may be written as

$$\partial\rho/\partial\tau + \partial u/\partial x + \partial v/\partial y = 0, \tag{7.1}$$

$$\partial u/\partial\tau = -\partial p/\partial x, \quad \partial v/\partial\tau = -\partial p/\partial y, \tag{7.2}$$

$$\partial p/\partial\tau = \partial\rho/\partial\tau, \tag{7.3}$$

where v is the perturbation velocity in the y-direction. These equations can be combined to yield the wave equation for p, viz.

$$\partial^2 p/\partial x^2 + \partial^2 p/\partial y^2 - \partial^2 p/\partial\tau^2 = 0. \tag{7.4}$$

The initial and boundary conditions are as follows:

$$\text{For} \quad \tau \leqslant 0, \quad p = \partial p/\partial\tau = 0. \tag{7.5}$$

On the disturbed shock front the set of equations (3.3) hold with q replaced by v and r by y. From these it follows that

$$u = (1/A)(p - B\bar{u} - C\bar{p}) \tag{7.6a}$$

and

$$\partial v/\partial\tau = (1/D)(\partial p/\partial y - \Lambda_{21}\partial\bar{u}/\partial y - \Lambda_{22}\partial\bar{p}/\partial y) + \Lambda_{41}\partial\bar{v}/\partial\tau. \tag{7.6b}$$

If $y = f(\chi)$ represents the upper surface of an aerofoil, behind the shock on the aerofoil

$$v_1 = (W+U)\, f'\{x + (W+U)t\},$$

or in non-dimensional form

$$v = v_1/a_1 = m_1 f'(x + m_1\tau).$$

Using the second momentum equation (7.2), it follows that

$$\text{at} \quad y = 0, \quad \partial p/\partial y = -m_1^2 f''(x + m_1\tau). \tag{7.7}$$

Also along the shock (where $x = m\tau$), on the aerofoil

$$v = m_1 f'\{(m + m_1)\tau\},$$

which together with shock relation (7.6*b*) yields

$$\text{at} \quad x = m\tau, \quad y = 0, \quad \partial p/\partial y = Dm_1(m+m_1)f''\{(m+m_1)\tau\} + \{\Lambda_{21}\partial\bar{u}/\partial y + \Lambda_{22}\partial\bar{p}/\partial y - D\Lambda_{41}\partial\bar{v}/\partial\tau\}. \tag{7.8}$$

At infinity we prescribe

$$\text{as} \quad x \to -\infty, \quad y \to +\infty, \quad p(x, y, \tau) \text{ and its derivatives} \to 0. \tag{7.9}$$

The disturbance field ahead of the shock can be expressed as

$$\left.\begin{aligned} \bar{u} &= u_2/V = -(k_1/\beta)f'\{(x+m_1\tau) - \beta y\}, \\ \bar{v} &= v_2/V = -\beta\bar{u}, \quad \bar{p} = p_2/\gamma P_0 = -(k_2/k_1)\bar{u}, \end{aligned}\right\} \tag{7.10}$$

where the constants k_1 and k_2 are the same as before.

We introduce new variables $(\bar{x}, y, \bar{\tau})$ related to the variables (x, y, τ) by the Lorentz transformation (5.1). The plane $\bar{x} = 0$ corresponds to the undisturbed plane of the shock $x = m\tau$, and the wave equation remains unchanged, viz.

$$\partial^2 p/\partial\bar{x}^2 + \partial^2 p/\partial y^2 - \partial^2 p/\partial\bar{\tau}^2 = 0. \tag{7.11}$$

The initial conditions are

$$\text{for} \quad \bar{\tau} \leqslant 0, \quad p = \partial p/\partial\bar{\tau} = 0. \tag{7.12}$$

The boundary conditions (7.9) and (7.7) become, for $\bar{\tau} > 0$, $\bar{x} < 0$, $y > 0$,

$$\text{as} \quad \bar{x} \to -\infty, \quad y \to +\infty, \quad p \text{ and its derivatives} \to 0, \tag{7.13}$$

and

$$\text{at} \quad y = 0, \quad \partial p/\partial y = A_0 f''\{\bar{a}(\bar{\tau} + \lambda_0\bar{x})\}. \tag{7.14}$$

Making use of the continuity equation (7.1) together with (7.3), the first of the momentum equations (7.2), the conditions at the shock (7.6) and the upstream perturbations (7.10), we deduce a differential condition in terms of p,

$$\text{at} \quad \bar{x} = 0, \quad y > 0, \quad -\frac{\partial^2 p}{\partial y^2} + 2m\frac{\partial^2 p}{\partial\bar{x}\,\partial\bar{\tau}} + \left(1 + \frac{1}{M^2}\right)\frac{\partial^2 p}{\partial\bar{\tau}^2} = \frac{K}{\bar{b}}\frac{\partial}{\partial\bar{\tau}}[f''\{\bar{a}(\bar{\tau} - \bar{b}y)\}]. \tag{7.15}$$

Also from (7.8) we obtain

$$\text{at} \quad \bar{x} = 0, \quad y = 0, \quad \partial p/\partial y = B_0 f''(\bar{a}\bar{\tau}). \tag{7.16}$$

The contents $A_0, \bar{a}, \lambda_0, \bar{b}, K$ and B_0 are the same as before. Now we must solve (7.11) subject to the initial and boundary conditions (7.12)–(7.16).

To this formulation first we apply the Laplace transform with respect to $\bar{\tau}$ as defined by (6.1) and then the Fourier cosine transform with respect to y as defined below

$$\left.\begin{aligned} v(\bar{x},y,s) &= L\{p(\bar{x},y,\bar{\tau})\},\\ w(\bar{x},\alpha,s) &= F_c\{v(\bar{x},y,s)\} = \int_0^\infty v(\bar{x},y,s)\cos\alpha y\,dy.\end{aligned}\right\} \tag{7.17}$$

The equation (7.11) and the conditions (7.12) and (7.13) give

$$\partial^2 w/\partial\bar{x}^2 - \lambda^2 w = A_0 \exp\{s\lambda_0\bar{x}\}G(s), \tag{7.18}$$

and

$$\text{as} \quad \bar{x}\to-\infty, \quad w\to 0. \tag{7.19}$$

The condition at the shock (7.15), together with (7.16), yields at $\bar{x}=0$

$$(\lambda^2+s^2/M^2)w + 2ms\,\partial w/\partial\bar{x} = -B_0G(s) + KG(s)\,s^2/(\alpha^2+\bar{b}^2s^2), \tag{7.20}$$

where $\lambda^2 = \alpha^2+s^2$ and $G(s) = L\{f''(\bar{a}\bar{\tau})\}$. The formulation (7.18)–(7.20) is the same as obtained earlier (6.8)–(6.10), the solution of which is given by (6.14). The inversion of (6.14) in this case yields

$$\begin{aligned} p(\bar{x},y,\bar{\tau}) &= L^{-1}[F_c^{-1}\{w(\bar{x},\alpha,s)\}]\\ &= \frac{1}{\pi}\left(-A_0\int_0^{\bar{\tau}}d\mu\int_{[x-\{(\bar{\tau}-\mu)^2-y^2\}^{\frac{1}{2}}]}^{0}\frac{f''\{\bar{a}(\mu+\lambda_0\xi)\}}{\{(\bar{\tau}-\mu)^2-(\xi-\bar{x})^2-y^2\}^{\frac{1}{2}}}d\xi\right.\\ &\quad\left.+\sum_{i=1}^{5}A_i\int_0^{\bar{\tau}}d\mu\int_0^{[x+\{(\bar{\tau}-\mu)^2-y^2\}^{\frac{1}{2}}]}\frac{f''\{\bar{a}(\mu-\lambda_i\xi)\}}{\{(\bar{\tau}-\mu)^2-(\bar{x}-\xi)^2-y^2\}^{\frac{1}{2}}}d\xi\right). \end{aligned} \tag{7.21}$$

8. Applications

8.1. *Slender conical projectile*

Specializing the results of §6 to a slender conical projectile of semi-vertex angle ϵ, we obtain from (6.16)

$$\begin{aligned} p(\bar{x},r,\bar{\tau}) &= \tfrac{1}{2}\epsilon^2\left(-A_0\int_{\omega_0}^{0}\frac{d\xi}{\{(\bar{x}-\xi)^2+r^2\}^{\frac{1}{2}}}+\sum_{i=1}^{5}A_i\int_0^{\omega_i}\frac{d\xi}{\{(\bar{x}-\xi)^2+r^2\}^{\frac{1}{2}}}\right)\\ &= \tfrac{1}{2}\epsilon^2\left\{Q\sinh^{-1}\left(\frac{\bar{x}}{r}\right)-\sum_{i=0}^{5}A_i\sinh^{-1}\left(\frac{\bar{x}-\omega_i}{r}\right)\right\}, \end{aligned} \tag{8.1}$$

where

$$Q = \sum_{i=0}^{5}A_i = 2(A_0-B_0).$$

Transforming to the original variables (x,r,τ) and expressing the results in terms of conical variables defined by $\sigma = x/\tau$ and $\eta = r/\tau$, we obtain

$$p(\sigma,\eta) = \tfrac{1}{2}\epsilon^2\left\{Q\sinh^{-1}\left(\frac{\sigma-m}{(1-m^2)^{\frac{1}{2}}\eta}\right)-\sum_{i=0}^{5}A_i\sinh^{-1}\left(\frac{\sigma-m-\Omega_i}{(1-m^2)^{\frac{1}{2}}\eta}\right)\right\}, \tag{8.2}$$

where

$$\Omega_0 = \frac{1+\gamma_0 m}{1-\gamma_0^2}[(\sigma+\gamma_0)-\{(1+\gamma_0\sigma)^2-(1-\gamma_0^2)\eta^2\}^{\frac{1}{2}}],$$

and

$$\Omega_i = \frac{1-\gamma_i m}{1-\gamma_i^2}[(\sigma-\gamma_i)+\{(1-\gamma_i\sigma)^2-(1-\gamma_i^2)\eta^2\}^{\frac{1}{2}}],$$

with

$$\gamma_0 = (\lambda_0-m)/(1-\lambda_0 m) \quad \text{and} \quad \gamma_i = (\lambda_i+m)/(1+\lambda_i m),$$

for $i = 1, 2 \ldots 5$.

The expression for density variation (6.17) becomes

$$\rho = p(\sigma,\eta)+(E-1)p(m,\eta')-(Fk_1-Gk_2)\epsilon^2\cosh^{-1}(\sigma/\bar{c}\eta), \tag{8.3}$$

where $$p(m,\eta') = \tfrac{1}{2}\epsilon^2\sum_{i=0}^{5} A_i \sinh^{-1}\{\Omega_i(m,\eta')/(1-m^2)^{\frac{1}{2}}\eta'\}, \quad \eta' = m\eta/\sigma.$$

We see that $p(m,\eta')$ vanishes for $\sigma(1-m^2)^{\frac{1}{2}} \leqslant m\eta$, i.e. to the left of line OB (figure 2) and the third term in (8.3) vanishes for $\sigma \leqslant \bar{c}\eta$, i.e. to the left of line OI. These lines OB and OI are thus contact surfaces, which can be clearly seen in the shadowgraph shown by Blankenship & Merritt (1966). The rotationality of the flow is thereby restricted to the domain IOF.

At the surface of the conical projectile $\eta = \epsilon(m_1+\sigma)$, Ω_0 and Ω_i approximate to

$$\Omega_0 \approx -(1-\sigma)(1+\gamma_0 m)/(1+\gamma_0), \quad \Omega_i \approx (1+\sigma)(1-\gamma_i m)/(1+\gamma_i),$$

and the expression (8.2) simplifies to

$$p = \tfrac{1}{2}\epsilon^2\left\{-Q\sinh^{-1}\left(\frac{1}{\epsilon(1-m^2)^{\frac{1}{2}}}\frac{m-\sigma}{m_1+\sigma}\right) - A_0\sinh^{-1}\left(\frac{1}{\epsilon(1-m^2)^{\frac{1}{2}}}\frac{1-m}{1+\gamma_0}\frac{1}{m_1}\right)\right.$$
$$\left. + \sum_{i=1}^{5} A_i\sinh^{-1}\left(\frac{1}{\epsilon(1-m^2)^{\frac{1}{2}}}\frac{1+m}{1+\gamma_i}\frac{1-\gamma_i\sigma}{m_1+\sigma}\right)\right\}. \tag{8.4}$$

8.2. *Two-dimensional wedge*

Specializing the result (7.21) to a two-dimensional wedge of semi-angle ϵ, we have

$$p(\bar{x},y,\bar{\tau}) = -\frac{\epsilon}{\pi\bar{a}}\left\{\frac{A_0}{(1-\lambda_0^2)^{\frac{1}{2}}}\cos^{-1}\left(\frac{\bar{x}+\lambda_0\bar{\tau}}{\{(\bar{\tau}+\lambda_0\bar{x})^2-(1-\lambda_0^2)y^2\}^{\frac{1}{2}}}\right)\right.$$
$$\left. -\sum_{i=1}^{5}\frac{A_i}{(1-\lambda_i^2)^{\frac{1}{2}}}\cos^{-1}\left(\frac{\lambda_i\bar{\tau}-\bar{x}}{\{(\bar{\tau}-\lambda_i\bar{x})^2-(1-\lambda_i^2)y^2\}^{\frac{1}{2}}}\right)\right\}, \tag{8.5}$$

where λ_0 and the λ_i's are <1. Changing to the original variables (x,y,τ) and expressing the result in terms of conical variables $\sigma = x/\tau$, $\eta = y/\tau$

$$p(\sigma,\eta) = -\frac{\epsilon}{\pi\bar{a}(1-m^2)^{\frac{1}{2}}}\left\{\bar{A}_0\cos^{-1}\left(\frac{\gamma_0+\sigma}{\{(1+\gamma_0\sigma)^2-(1-\gamma_0^2)\eta^2\}^{\frac{1}{2}}}\right)\right.$$
$$\left. -\sum_{i=1}^{5}\bar{A}_i\cos^{-1}\left(\frac{\gamma_i-\sigma}{\{(1-\gamma_i\sigma)^2-(1-\gamma_i^2)\eta^2\}^{\frac{1}{2}}}\right)\right\}, \tag{8.6}$$

where $$\bar{A}_0 = A_0(1+\gamma_0 m)/(1-\gamma_0^2)^{\frac{1}{2}}, \quad \bar{A}_i = A_i(1-\gamma_i m)/(1-\gamma_i^2)^{\frac{1}{2}}.$$

The expressions for the pressure on the face of the wedge ($\eta = 0$) and along the shock ($\sigma = m$) can be shown by laborious calculation to be the same as obtained by Smyrl (1963).

9. Numerical results and discussion

Pressure distributions on the surface of the cone have been calculated for the Mach-reflexion region E to F. We consider a cone of semi-vertex angle $\epsilon = 0{\cdot}025$ with $M' = 1{\cdot}5$ and $2{\cdot}5$ for various values of M and with $M = 2{\cdot}0$ and $4{\cdot}0$ for various values of M', thus demonstrating the effect of shock strength and the cone speed respectively. The results are illustrated in figures 5 and 6. The scale is adjusted to make the distance EF the same in all cases. In figure 7, a comparison is drawn between our results and those obtained by Blankenship (1965) by

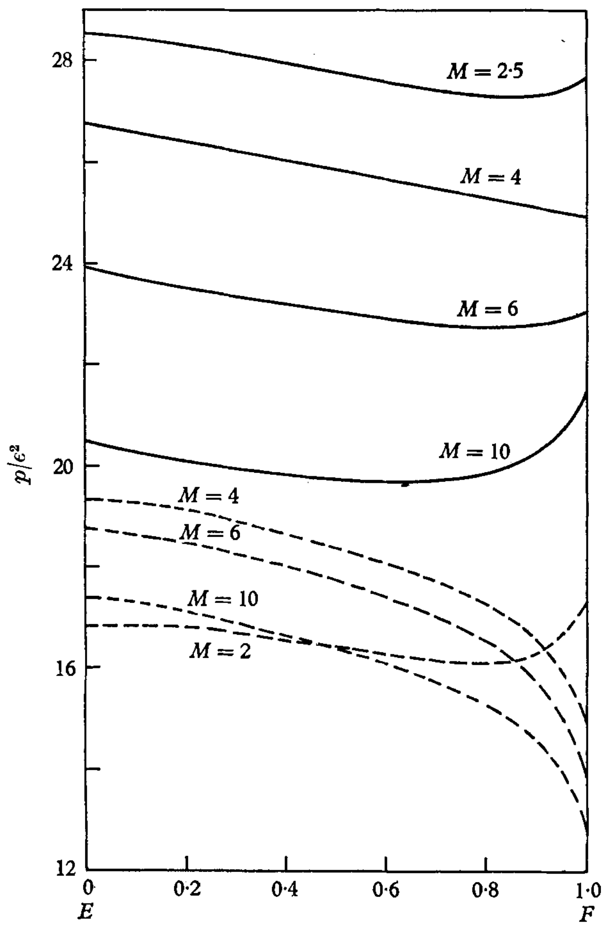

FIGURE 5. Pressure distribution on the cone surface for $\epsilon = 0{\cdot}025$ for various values of M. ——, $M' = 2{\cdot}5$; – – –, $M' = 1{\cdot}5$.

numerical methods for $\epsilon = 0{\cdot}025$, $M' = 2{\cdot}5$ and $M = 11{\cdot}25$ and $6{\cdot}25$. Two features emerge from it. First, the starting values of the pressure at E are not the same in the two cases, Blankenship's values being higher. This may be due to a numerical mistake in the latter's results. Since Blankenship starts from the well-known conical solution in the region AEC, while the author obtains the same conical solution in this region from his analysis, the values of pressure at E should have been the same in the two cases. Secondly, the behaviour of the curves given by the two procedures is different. The discrepancy in the behaviour can be attributed to the fact that Blankenship considers the value of $r(\partial p/\partial r)$ on the cone surface to be constant from E to F (figure 2), which in fact is the case but he does not consider any jump in $r(\partial p/\partial r)$ at F. While in our formulation when the shock is approached along the body and when the body is approached along the shock, the two limits of $r(\partial p/\partial r)$ are different at F, thereby providing a jump in $r(\partial p/\partial r)$, though the assumption that the flow is tangential

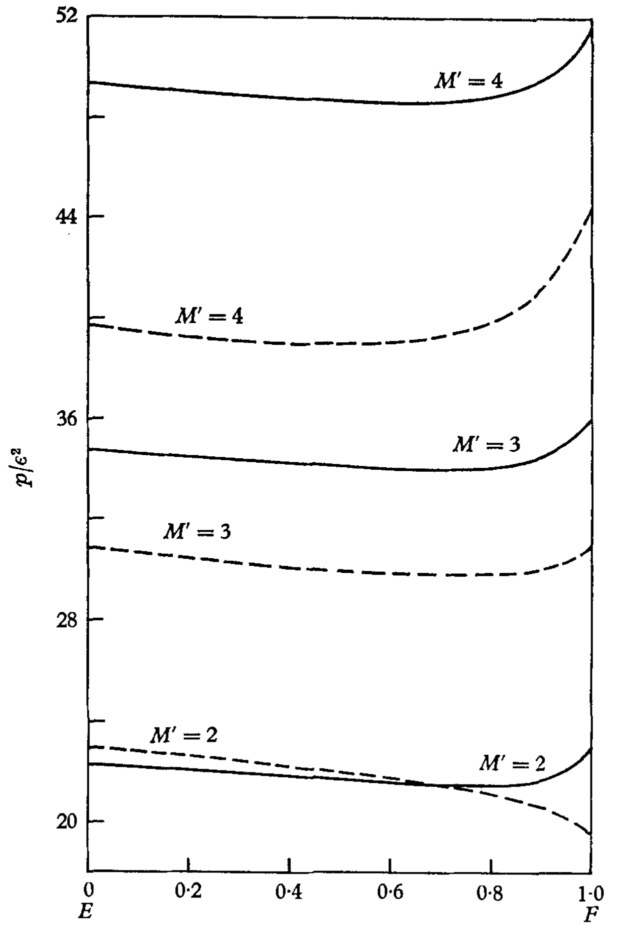

FIGURE 6. Pressure distribution on the cone surface for $\epsilon = 0{\cdot}025$ for various values of M'. ——, $M = 2{\cdot}0$; – – –, $M = 4{\cdot}0$.

to the body even at the root of the shock still holds. Incidentally, if we consider the value of $r(\partial p/\partial r)$ constant along the cone surface from E to F without any jump (i.e. B_0 is taken as A_0 in the analysis), the behaviour of the results thus obtained for the pressure on the cone surface is closer to that obtained by Blankenship. These results are also plotted alongside the other results in figure 7.

The complete pressure field in the Mach-reflexion region is plotted in figure 8 for $\epsilon = 0{\cdot}025$, $M' = 2{\cdot}5$ and $M = 11{\cdot}25$. This allows fuller comparison with Blankenship (1965, figure 6). The isobars in the two cases agree fairly well, except in the vicinity of the cone surface as can be expected from the preceding discussion.

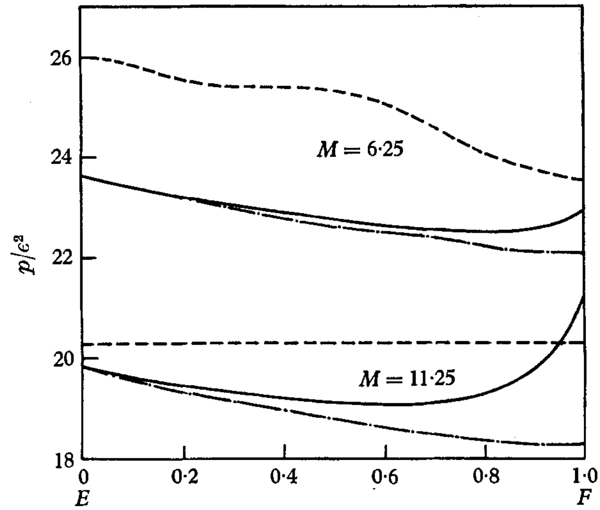

FIGURE 7. Comparison of results—pressure variation on the cone surface for $\epsilon = 0{\cdot}025$, $M' = 2{\cdot}5$, $M = 11{\cdot}25$ and $6{\cdot}25$; ——, present theory; —·—, present theory with B_0 taken as A_0; - - -, Blankenship (1965).

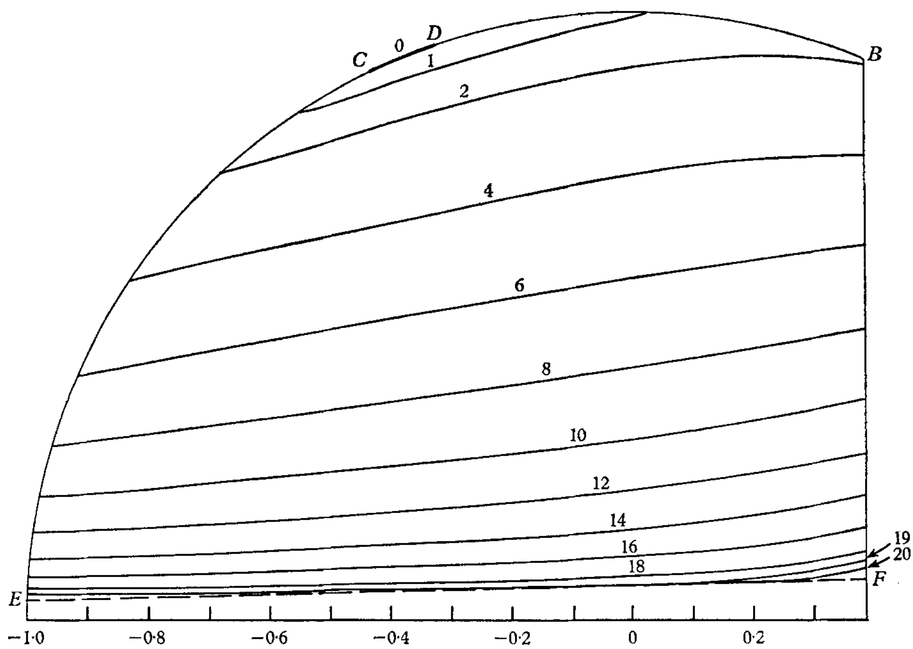

FIGURE 8. Isobars (p/ϵ^2) for $\epsilon = 0{\cdot}025$, $M' = 2{\cdot}5$, $M = 11{\cdot}25$.

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